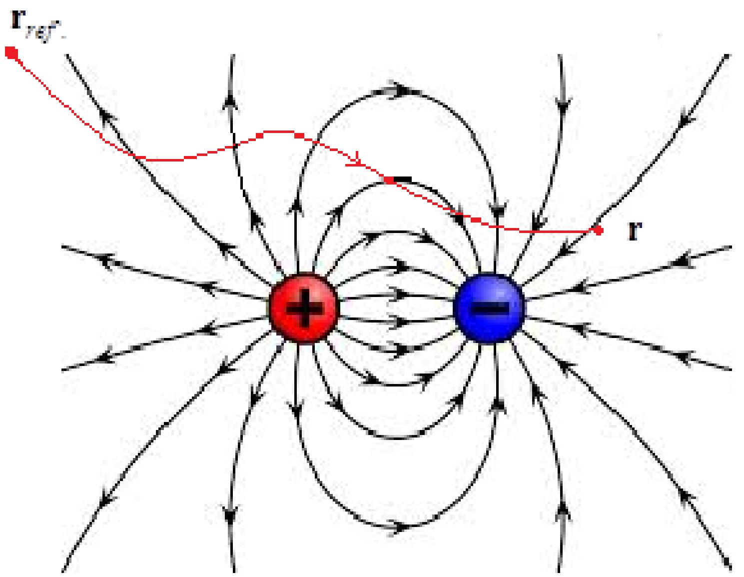
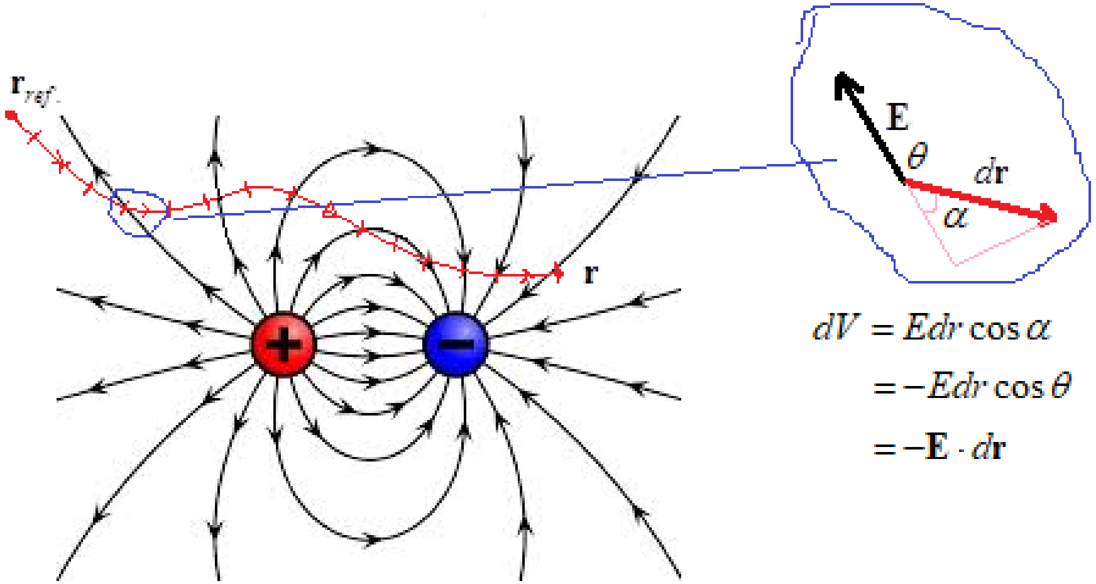
A.3 Electric Potential

Want to introduce another property of electric fields, quantified in terms of a concept called the electric potential. It will provide further insight into the structure of an electric field surrounding a set of charges, and provide another means of calculating these fields.





$$\begin{aligned} dV &= E dr \cos\alpha \\ &= -E dr \cos\theta \\ &= -\mathbf{E}\cdot d\mathbf{r} \end{aligned}$$

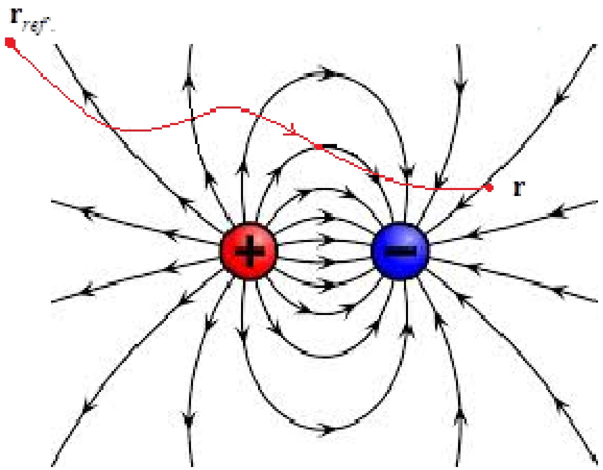
The electric potential at point r, symbolized V(r), can be thought of as the ‘amount of electric field lines you travel against going from $r_{reference}$ to r’. $r_{reference}$ is usually taken to be infinitely far away from the charges.

Have to quantify ‘amount of electric field lines you travel against’. This is done for a typical segment of the path above. And total result is:

$$V(\mathbf{r}) = -\int_{r_{ref.}}^{r} \mathbf{E}\cdot d\mathbf{r} \qquad \text{Electric potential}$$

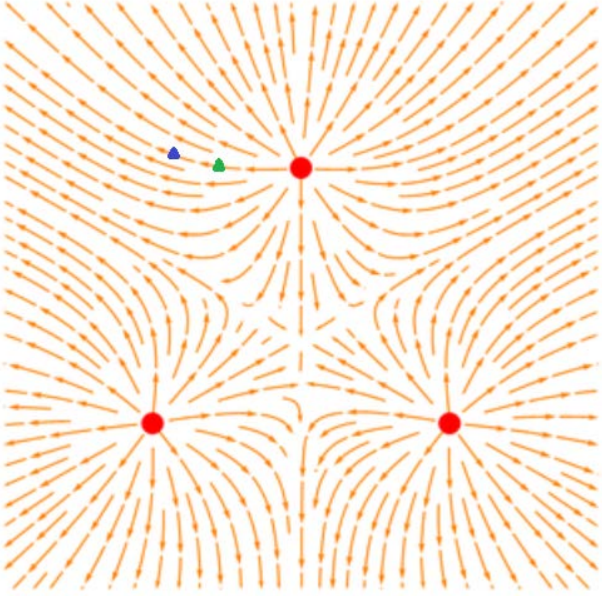
$$\text{units} = \frac{\text{N}}{\text{C}}\cdot\text{m} = \frac{\text{N}\cdot\text{m}}{\text{C}} = \frac{\text{J}}{\text{C}} \equiv \text{Volt (V)}$$

A.3 Electric Potential



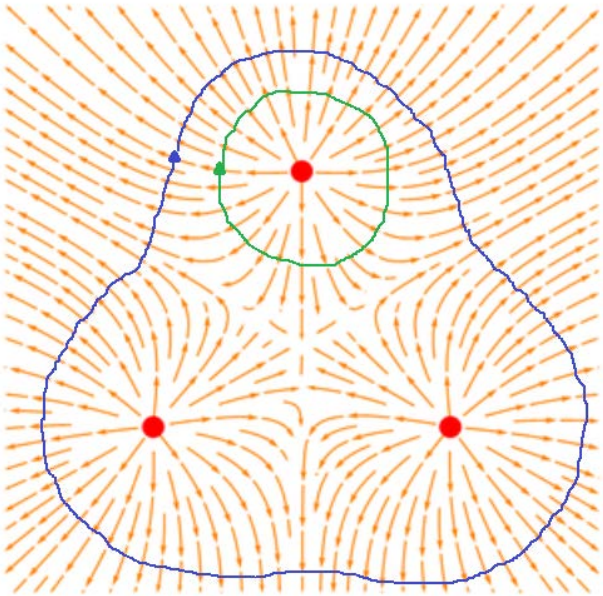
Salient Properties of the Electric Potential V(r)

- V(r) doesn't depend on the path one takes to get from $r_{ref.}$ to r, and so every point in space has a unique potential, so V(r) is truly a 'function'.
- V(r) increases as r moves against field lines.
- V(r) decreases as r moves with field lines.
- V(r) remains constant as r moves perpendicular to field lines. The set of r's at the same potential is called an 'equipotential'.



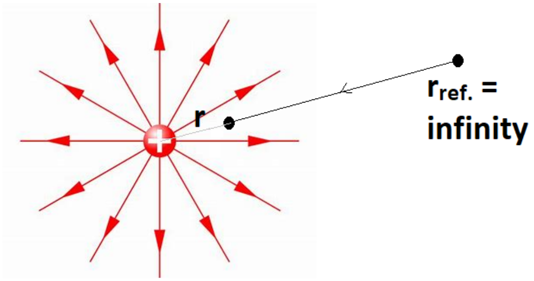
For instance, consider the two points in the Electric field shown. Draw the equipotential running through each point. Which is at the higher potential?

V_{green} is higher than V_{blue}



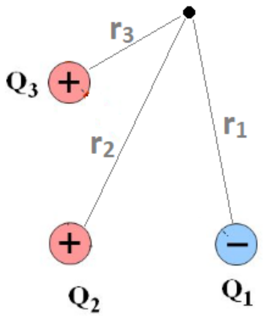
A.3 Electric Potential

Now we want to practice getting the potential for different charge configurations. First we will need to know how to calculate the potential of a point charge, q.



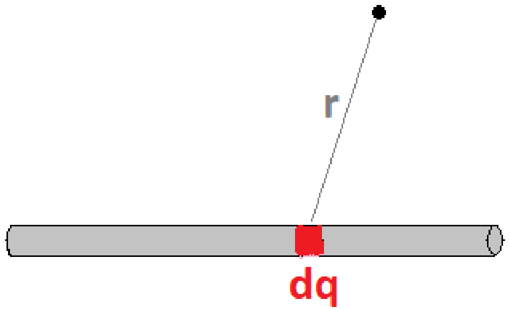
$$V = -\int_{r_{ref}}^{r} \mathbf{E} \cdot d\mathbf{r} = -\int_{\infty}^{r} \frac{kq}{r^2} \hat{\mathbf{r}} \cdot d\mathbf{r} = -\int_{\infty}^{r} \frac{kq}{r^2} dr = \frac{kq}{r} \longrightarrow \boxed{V = \frac{kq}{r}}$$

Note that q here is *not* absolute value, so if charge is negative, then you should put the negative sign in.



If we have multiple charges, then we just add up the potentials of each charge. Note there are no vectors going on here. It is a 'scalar' sum.

$$V = \sum V_i \qquad V_i = \frac{kq_i}{r_i}$$

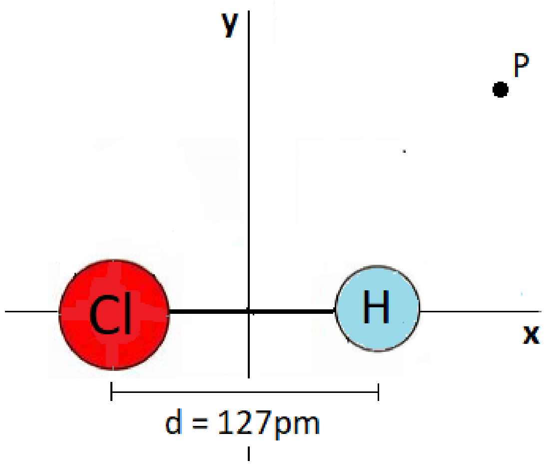


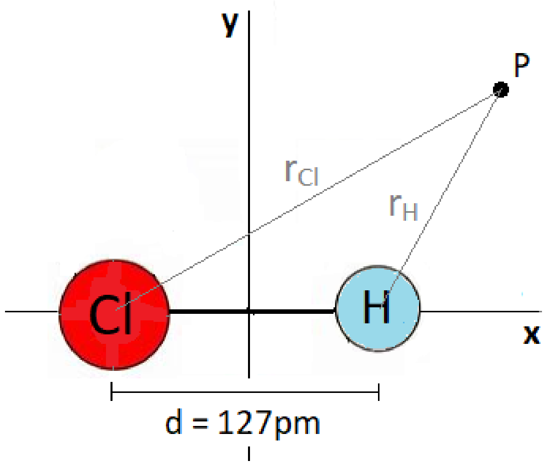
If we have a continuous distribution of charge, then we add up the potentials of each infinitesimal charge, dq. Again, no vectors to add here. It's a 'scalar' integral.

$$V = \int dV \qquad dV = \frac{kdq}{r}$$

A.3 Electric Potential

Consider the HCl molecule again, and coordinate P = (100pm, 100pm).
What is the electric potential at that point?

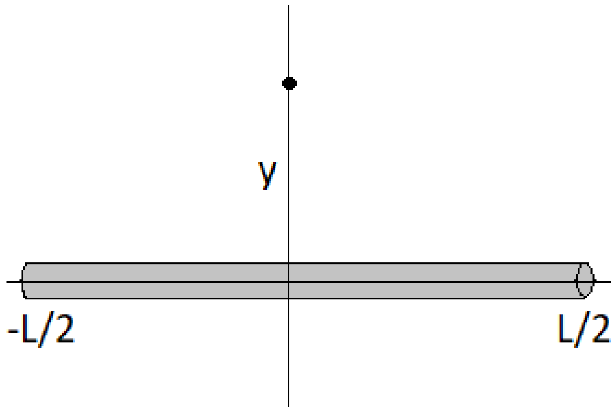


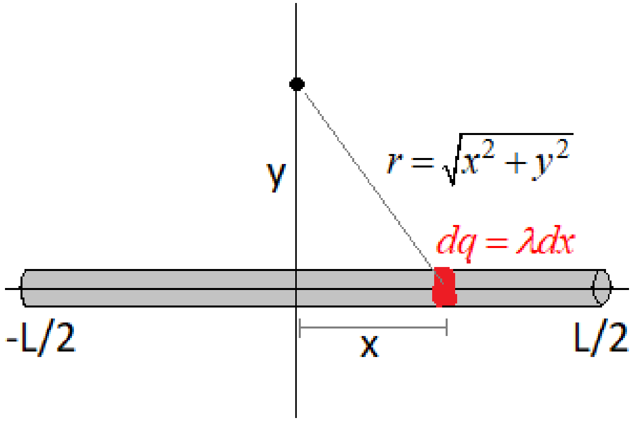


$$
\begin{aligned}
V &= V_{Cl} + V_H \\
&= \frac{kq_{Cl}}{r_{Cl}} + \frac{kq_H}{r_H} \\
&= \frac{(9\times10^9)(-1.6\times10^{-19})}{\sqrt{(100\times10^{-12}+63.5\times10^{-12})^2+(100\times10^{-12})^2}} + \frac{(9\times10^9)(1.6\times10^{-19})}{\sqrt{(100\times10^{-12}-63.5\times10^{-12})^2+(100\times10^{-12})^2}} \\
&= 6.01\text{V}
\end{aligned}
$$

A.3 Electric Potential

Or consider a 50cm rod centered on the y-axis, with charge q = 4nC evenly distributed on it.   What is the the potential at the point y = 30cm?



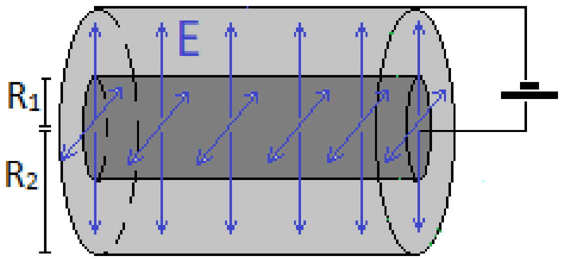


$$dV=\frac{kdq}{r}=\frac{k(\lambda dx)}{\sqrt{x^2+y^2}} \qquad \lambda=\frac{q}{L}$$

$$\begin{aligned}
V(y)&=\int dV\\
&=\int_{-L/2}^{L/2}\frac{k\lambda dx}{\sqrt{x^2+y^2}}\\
&=k\lambda\ln\left[x+\sqrt{x^2+y^2}\right]\Bigg|_{x=-L/2}^{x=L/2}\\
&=k\lambda\ln\left[\frac{L}{2}+\sqrt{\left(\frac{L}{2}\right)^2+y^2}\right]-k\lambda\ln\left[-\frac{L}{2}+\sqrt{\left(-\frac{L}{2}\right)^2+y^2}\right]\\
&=\frac{kq}{L}\ln\left[\frac{L/2+\sqrt{(L/2)^2+y^2}}{-L/2+\sqrt{(L/2)^2+y^2}}\right]\\
&=109\,\mathrm{V}
\end{aligned}$$

A.3 Electric Potential

Or let's revisit the Geiger counter. We said that its electric field was all that stuff down there. What is the potential of the inner cylinder, w/r to the outer cylinder?



$$E = \begin{cases} \dfrac{C}{4\varepsilon_0} r^3 & r < R_1 \\ \dfrac{C}{4\varepsilon_0} \dfrac{R_1^4}{r} & R_1 < r < R_2 \\ 0 & R_2 < r \end{cases}$$

If we know the field, then it's perhaps easiest to get V by direct integration. The potential difference can be found by integrating dV between the outer plate and inner plate….

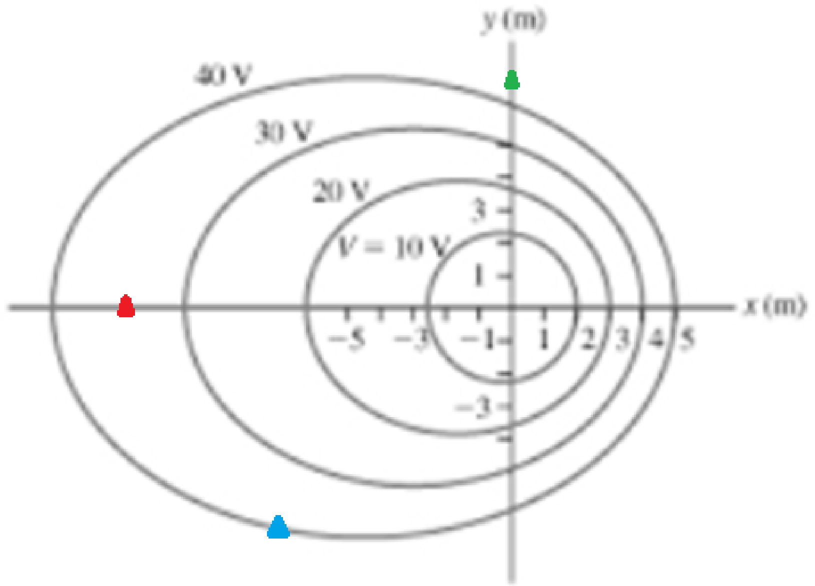
$$\Delta V = -\int_{R_2}^{R_1} \mathbf{E} \cdot d\mathbf{r} \quad = -\int_{R_2}^{R_1} \frac{C}{4\varepsilon_0} \frac{R_1^4}{r} dr$$

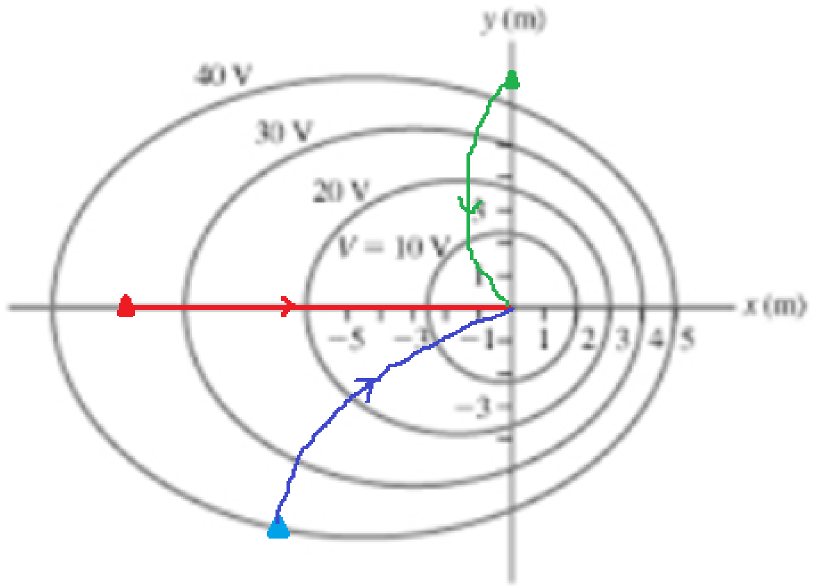
$$= -\frac{C}{4\varepsilon_0} R_1^4 \ln r \Big|_{r=R_2}^{r=R_1} = -\frac{C}{4\varepsilon_0} R_1^4 \ln\left(\frac{R_1}{R_2}\right)$$

$$= \frac{C}{4\varepsilon_0} R_1^4 \ln\left(\frac{R_2}{R_1}\right)$$

A.3 Electric Potential

From the electric field, we can get the potential. We can also go backwards. Consider following example, which we'll use to segue into our main goal, which is to work out how to construct **E**, given knowledge of V(**r**).

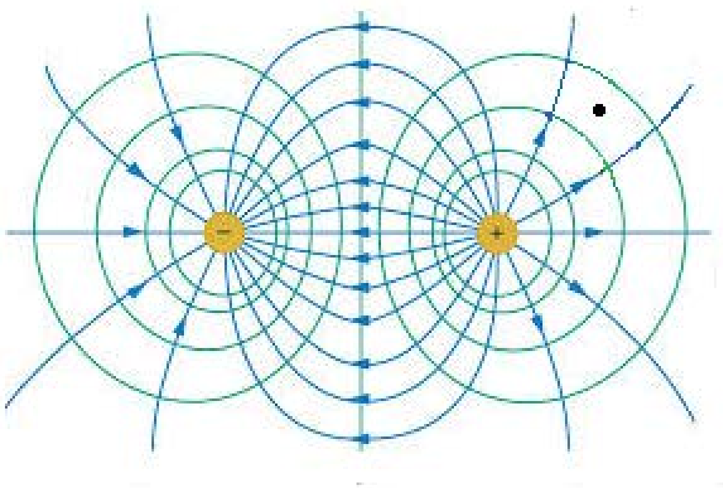




Consider the equipotentials above. Draw the three electric field lines that go through the indicated points. What's the sign of the charge(s) at the origin?

Field lines run perpendicular to the equipotentials, and towards lower potential. Charge at origin must be a *negative*, since field lines are going *towards* it.

A.3 Electric Potential

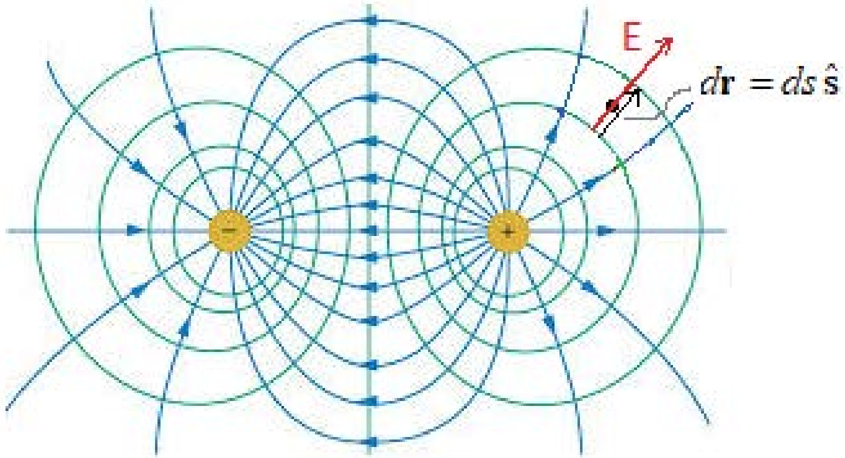


So suppose we know the potential everywhere. How do we get the field, say at that point in space? We know it points perpendicular to the equipotentials, from higher to lower - another way to say this is, that it points in the direction along which the potential most rapidly *decreases*. Let this direction be called $\hat{\boldsymbol{s}}$. And let the distance between the two equipotentials *along this direction* be ds. Make d**r** the associated displacement vector. Then finally, the potential difference, dV, between the two equipotentials would satisfy:

$$dV = -\mathbf{E}\cdot d\mathbf{r} = -\mathbf{E}\cdot ds\hat{\mathbf{s}} = -Eds\cos 0^\circ = -Eds$$

$$\rightarrow E = -\frac{dV}{ds} = \left|\frac{dV}{ds}\right|$$

$$\boxed{\mathbf{E} = \left|\frac{dV}{ds}\right|\hat{\mathbf{s}}}$$



Another way to do it, if we just use the generic identity, $d\boldsymbol{r} = dx\hat{\boldsymbol{\imath}} + dy\hat{\boldsymbol{\jmath}} + dz\hat{\boldsymbol{k}}$:

$$dV = -\mathbf{E}\cdot d\mathbf{r} = -E_x dx - E_y dy - E_z dz$$

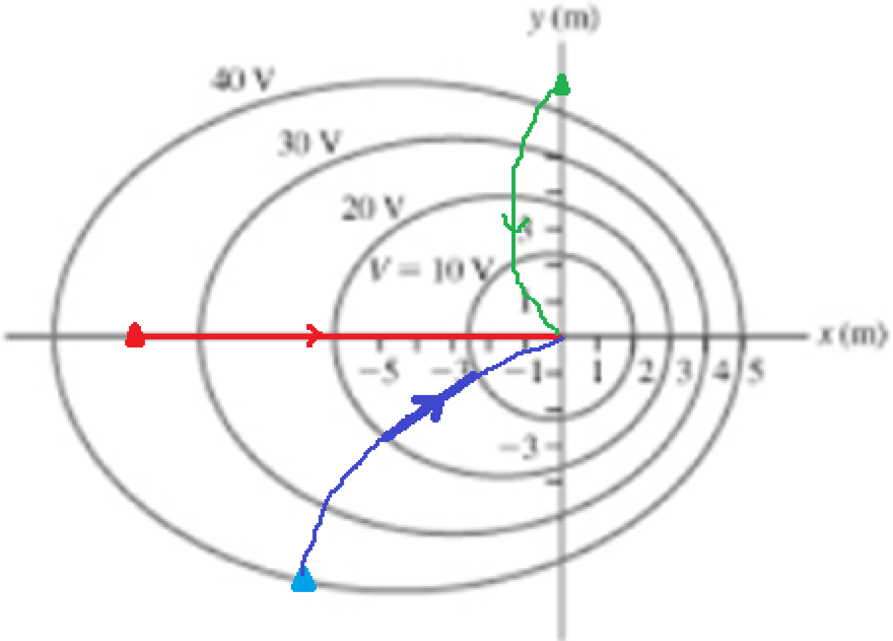
$$\rightarrow E_x = -\frac{dV}{dx},\ E_y = -\frac{dV}{dy},\ E_z = -\frac{dV}{dz}$$

$$\rightarrow \mathbf{E} = -\frac{dV}{dx}\hat{\mathbf{i}} - \frac{dV}{dy}\hat{\mathbf{j}} - \frac{dV}{dz}\hat{\mathbf{k}}$$

$$\boxed{\mathbf{E} = -\nabla V}$$

$$\rightarrow \mathbf{E} = -\nabla V \qquad \nabla = \frac{d(\ )}{dx}\hat{\mathbf{i}} + \frac{d(\ )}{dy}\hat{\mathbf{j}} + \frac{d(\ )}{dz}\hat{\mathbf{k}}$$

A.3 Electric Potential



On the graph, estimate the electric field strength along the highlighted (blue) field line segment.

$$E = \left|\frac{dV}{ds}\right|$$

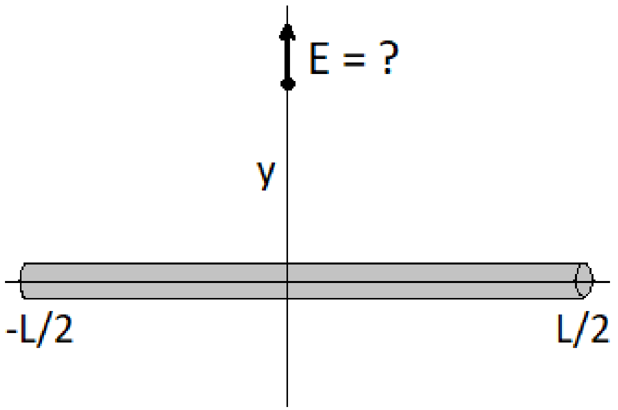
$$\approx \left|\frac{10\text{V} - 20\text{V}}{2.5\text{m}}\right|$$

$$\approx 4\frac{\text{V}}{\text{m}} \qquad \left(\frac{\text{V}}{\text{m}} = \frac{\text{N}}{\text{C}}\right)$$

Along which of the curves is the electric field generally the highest?

Along the green curve, as the equipotentials are more closely spaced there.

A.3 Electric Potential



In a previous problem, we derived the following expression for the electric potential along an axis running through the center of a wire. Infer from this expression the y-component of the electric field along the same axis.

$$V(y)=\frac{kq}{L}\ln\left[\frac{L/2+\sqrt{(L/2)^2+y^2}}{-L/2+\sqrt{(L/2)^2+y^2}}\right]$$

$$E_y=-\frac{dV}{dy}\hat{\mathbf{j}}$$

$$=-\frac{d}{dy}\left\{\frac{kq}{L}\ln\left[\frac{L/2+\sqrt{(L/2)^2+y^2}}{-L/2+\sqrt{(L/2)^2+y^2}}\right]\right\}\hat{\mathbf{j}}=-\frac{kq}{L}\frac{d}{dy}\left\{\ln(L/2+\sqrt{(L/2)^2+y^2})-\ln(-L/2+\sqrt{(L/2)^2+y^2})\right\}\hat{\mathbf{j}}$$

$$=-\frac{kq}{L}\left\{\frac{\frac{y}{\sqrt{(L/2)^2+y^2}}}{L/2+\sqrt{(L/2)^2+y^2}}-\frac{\frac{y}{\sqrt{(L/2)^2+y^2}}}{-L/2+\sqrt{(L/2)^2+y^2}}\right\}\hat{\mathbf{j}}=-\frac{kq}{L}\frac{y}{\sqrt{(L/2)^2+y^2}}\left\{\frac{1}{L/2+\sqrt{(L/2)^2+y^2}}-\frac{1}{-L/2+\sqrt{(L/2)^2+y^2}}\right\}\hat{\mathbf{j}}$$

$$=-\frac{kq}{L}\frac{y}{\sqrt{(L/2)^2+y^2}}\left\{\frac{-L}{(L/2)^2+y^2-(L/2)^2}\right\}\hat{\mathbf{j}}=\frac{kq}{y\sqrt{(L/2)^2+y^2}}\hat{\mathbf{j}}$$

A.3 Electric Potential

Last one….suppose you were given some random function for V(x,y,z). Calculate is the magnitude and direction of the electric field at the coordinate (-5,-5,5)?

$$V(x,y,z) = 50\left(\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{4^2}\right)(\mathrm{V})$$

$$\mathbf{E}(x,y,z) = -\nabla V(x,y,z)$$

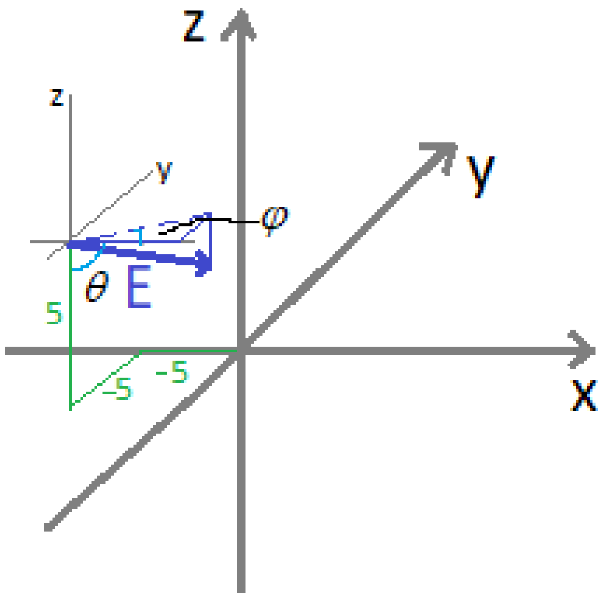
$$= -\left(\frac{d(\ )}{dx}\hat{\mathbf{i}}+\frac{d(\ )}{dy}\hat{\mathbf{j}}+\frac{d(\ )}{dz}\hat{\mathbf{k}}\right)\left[50\left(\frac{x^2}{2^2}+\frac{y^2}{3^2}+\frac{z^2}{4^2}\right)\right]$$

$$= -\left(50\cdot\frac{2x}{2^2}\hat{\mathbf{i}}+50\cdot\frac{2y}{3^2}\hat{\mathbf{j}}+50\cdot\frac{2z}{4^2}\hat{\mathbf{k}}\right)$$

$$= -100\left(\frac{x}{2^2}\hat{\mathbf{i}}+\frac{y}{3^2}\hat{\mathbf{j}}+\frac{z}{4^2}\hat{\mathbf{k}}\right)$$

$$\mathbf{E}(-5,-5,5) = -100\left(\frac{-5}{2^2}\hat{\mathbf{i}}+\frac{-5}{3^2}\hat{\mathbf{j}}+\frac{5}{4^2}\hat{\mathbf{k}}\right)$$

$$= 125\hat{\mathbf{i}}+56\hat{\mathbf{j}}-31\hat{\mathbf{k}}$$



$$E = \sqrt{125^2+56^2+(-31)^2} = 140\,\frac{\mathrm{N}}{\mathrm{C}}$$

$$\theta = \cos^{-1}\left(\frac{E_z}{E}\right) = \cos^{-1}\left(\frac{-31}{140}\right) = 77^\circ$$

$$\varphi = \tan^{-1}\left(\frac{E_y}{E_x}\right) = \tan^{-1}\left(\frac{56}{125}\right) = 24^\circ$$